The Functional Variable Method to Some Complex Nonlinear Evolution Equations

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Abstract- **In this paper, it shows the applicability of the functional variable method for the exact solutions of some complex nonlinear evolution equations. By using this scheme, we found some exact solutions of the nonlinear Schrödinger, new Hamiltonian amplitude and coupled Higgs fields equations. Then, some of the solitary solutions are converted to periodic solutions or hyperbolic function solutions by a simple transformation.**

Keywords- The Functional Variable Method; Nonlinear Schrödinger Equation; New Hamiltonian Amplitude Equation; Coupled Higgs Fields Equations

I. INTRODUCTION

It is known that many phenomena in scientific. fields can be described by nonlinear partial differential equations. Interests in nonlinear wave equations have grown rapidly in recent years. This should not be surprising since nonlinear wave phenomena are observed in many physics areas, such as fluid mechanics, plasma physics, optical fibers, hydrodynamics, biology, solid state physics, etc. As is well known, nonlinear dynamical systems are often described by nonlinear partial differential equations as for example, the nonlinear Schrödinger equation, the Korteweg-de Vries equation, the Klein-Gordon equation, and many others. The finding of exact solutions for such nonlinear evolution equations (NLEEs) is very important to better understand the nonlinear phenomena. It should be noted that the key factors which determine the exact solution of a given NLEE are the dependent dispersive and nonlinear coefficients. These coefficients can be constants or variables depending on the nature of the considered medium which can be homogeneous or inhomogeneous.

The study of NLEEs has been going on for the past few decades [1-19]. There has been a tremendous improvement in this area during this period. Several NLEEs have been formulated depending on the physical situation. Besides, many such equations are generalized to study their general behavior so that the special cases are truly meaningful both from the physical and mathematical point of view.

In recent years, various powerful methods of integrability have been established and developed. These techniques are applied left and right to these various NLEEs to solve them and obtain closed form of solutions of physical relevance. There are various solutions that are obtained by incorporating these techniques of integrability. They are soliton solutions, travelling waves, cnoidal waves, kinks and anti-kinks, peakons, cuspons and stumpons just to name a few. The study of nonlinear PDEs mostly yields travelling wave solutions.

To achieve our goal, we organize the paper as follows. In Section 2, we describe functional variable method for finding exact solutions of complex nonlinear evolution equations. In Section 3 to Section 5, we illustrate this method in detail with the celebrated nonlinear Schrödinger, new Hamiltonian amplitude and coupled Higgs fields equations. Finally, some important conclusions are given.

II. THE FUNCTIONAL VARIABLE METHOD

Zerarka at all have summarized for using functional variable method [20]. For a given nonlinear partial differential equation (PDE), written in several independent variables as

$$P(u, u_t, u_x, u_y, u_z, u_{xy}, u_{yz}, ...) = 0 \qquad (2.1)$$

where the subscript denotes partial derivative, P is some function, and $u\{t, x, y, z, ...\}$ is called a dependent variable or unknown function to be determined.

We firstly introduce the new wave variable as:

$$\xi = k(x + y + ... + ct) \text{ or } \xi = x + y + ... - ct$$

The nonlinear partial differential equation can be converted to an ordinary differential equation (ODE) like:

$$Q(U, U', U'', U''', ...) = 0 \qquad (2.2)$$

Equation (2.2) is then integrated as long as all terms contain derivatives where integration constants are considered zeros.

Let us make a transformation in which the unknown function U considered as a functional variable in the form [21]

$$U_\xi = \frac{dU}{d\xi} = F(U) \qquad (2.3)$$

and some successively derivatives of U are,

$$U_{\xi\xi} = \frac{d^2U}{d\xi^2} = \frac{1}{2}(F^2)',$$

$$U_{\xi\xi\xi} = \frac{d^3U}{d\xi^3} = \frac{1}{2}(F^2)''\sqrt{F^2}, \qquad (2.4)$$

$$U_{\xi\xi\xi\xi} = \frac{d^4U}{d\xi^4} = \frac{1}{2}[(F^2)'''F^2 + (F^2)''(F^2)'],$$

where "'" stands for $\dfrac{d}{dU}$. The ODE (2.2) can be reduced in terms of U, F which are real or complex variable functions and using the expressions in (2.4) into (2.2) give

$$R(U, F, F'', F''', F^{(4)}, ...) = 0 \qquad (2.5)$$

The key idea of this particular form (2.5) is of special interest because it admits analytical solutions for a large class of nonlinear wave type equations. After integration, the Eq. (2.5) provides the expression of F, and this in turn together with (2.3) gives the appropriate solutions to the original problem [22].

In order to illustrate how the method works we examine some examples treated by other approaches. This matter is exposed in the following section.

III. NONLINEAR SCHRÖDINGER EQUATION

Consider the nonlinear Schrödinger equation [23-24]

$$iu_t + pu_{xx} + q|u|^2 u = 0 \qquad (3.1)$$

Exact solutions for the nonlinear Schrödinger equation have been shown by first integral method [25].

By making the transformation,

$$u(x,t) = e^{i\theta}U(\xi), \ \ \theta = \alpha x + \beta t, \ \ \xi = x - 2p\alpha t \qquad (3.2)$$

the Eq. (3.1) is carried to an ODE

$$-(\beta + \alpha^2 p)U + pU'' + qU^3 = 0 \qquad (3.3)$$

and then using the transformation

$$U_\xi = \frac{dU}{d\xi} = F(U), \qquad (3.4)$$

in Eq. (3.3) yields

$$-(\beta + \alpha^2 p)U - qU^3 = p\frac{(F(U)^2)'}{2} \qquad (3.5)$$

where the prime denotes differentiation with respect to ξ. Integrating (3.5) and after the mathematical manipulations, we have

$$F(U) = \sqrt{U^2\left(\frac{\beta + \alpha^2 p}{p} - \frac{qU^2}{2p}\right)}. \qquad (3.6)$$

Using Transformation (3.4) then we setting the constants of integration to ξ_0 and we can obtain the result below

$$\begin{aligned} U(\xi) &= \frac{\pm 1}{\sqrt{|c_1|}}\frac{1}{\cos(\sqrt{|c_2|}(\xi - \xi_0))}. \text{ if } c_1 < 0 \text{ and } c_2 < 0,\\ U(\xi) &= \frac{\pm 1}{\sqrt{|c_1|}}\frac{1}{\sin(\sqrt{|c_2|}(\xi - \xi_0))}. \text{ if } c_1 > 0 \text{ and } c_2 < 0,\\ U(\xi) &= \frac{\pm 1}{\sqrt{|c_1|}}\frac{1}{\cosh(\sqrt{|c_2|}(\xi - \xi_0))}. \text{ if } c_1 > 0 \text{ and } c_2 > 0,\\ U(\xi) &= \frac{\pm 1}{\sqrt{|c_1|}}\frac{1}{\sinh(\sqrt{|c_2|}(\xi - \xi_0))}. \text{ if } c_1 < 0 \text{ and } c_2 > 0. \end{aligned} \qquad (3.7)$$

Where $c_1 = \dfrac{q}{2(\beta + \alpha^2 p)}$ and $c_2 = \dfrac{\beta + \alpha^2 p}{p}$. We can easily obtain following exact solutions (periodic solutions or hyperbolic function solutions).

$$u_1(x,t) = -e^{i(\alpha x + \beta t)}\sqrt{\frac{2(\beta + \alpha^2 p)}{q}}\, i\,\mathrm{csc}h\left(\sqrt{\frac{\beta + \alpha^2 p}{p}}(x - 2p\alpha t)\right) \qquad (3.8)$$

$$u_2(x,t) = e^{i(\alpha x + \beta t)}\sqrt{\frac{2(\beta + \alpha^2 p)}{q}}\csc\left(\sqrt{-\frac{\beta + \alpha^2 p}{p}}(x - 2p\alpha t)\right) \qquad (3.9)$$

$$u_3(x,t) = e^{i(\alpha x + \beta t)}\sqrt{\frac{2(\beta + \alpha^2 p)}{q}}\,\mathrm{sec}h\left(\sqrt{\frac{\beta + \alpha^2 p}{p}}(x - 2p\alpha t)\right) \qquad (3.10)$$

$$u_4(x,t) = e^{i(\alpha x + \beta t)}\sqrt{-\frac{2(\beta + \alpha^2 p)}{q}}\sec\left(\sqrt{-\frac{\beta + \alpha^2 p}{p}}(x - 2p\alpha t)\right) \qquad (3.11)$$

Comparing our results and Ma's results [23], Zhou's results [24] with Taghizadeh's results [25] then it can be seen that the results are the same.

IV. NEW HAMILTONIAN AMPLITUDE EQUATION

Let us consider a new Hamiltonian amplitude equation [26, 27]:

$$iu_x + u_{tt} + 2\sigma|u|^2 u - \varepsilon u_{xt} = 0, \qquad (4.1)$$

where $\sigma = \pm 1$, $\varepsilon \ll 1$ was recently introduced in [28]. This is an equation which governs certain instabilities of modulated wave trains, with the additional term $-\varepsilon u_{xt}$ overcoming the ill-posedness of the unstable nonlinear Schrödinger equation. It is a Hamiltonian analogue of the Kuramoto-Sivashinski equation which arises in dissipative systems and is apparently not integrable.

We use the transformation,

$$u(x,t) = e^{i\theta}U(\xi), \ \ \theta = \alpha x - \beta t, \ \ \xi = k(x - \lambda t),$$

the Eq. (4.1) becomes

$$k^2(\lambda^2 + \varepsilon\lambda)U''(\xi) + ik(1 + 2\beta\lambda + \varepsilon\alpha\lambda + \varepsilon\beta)U'(\xi) - (\alpha + \beta^2 + \varepsilon\alpha\beta)U(\xi) + 2\sigma(U(\xi))^3 = 0, \qquad (4.2)$$

where α, β, k and λ are constants and $U(\xi)$ is real function. If we take

$$\lambda = \frac{1 + \varepsilon\beta}{2\beta + \varepsilon\alpha}$$

Eq. (4.2) is transformed into

$$k^2(\lambda^2 + \varepsilon\lambda)U''(\xi) - (\alpha + \beta^2 + \varepsilon\alpha\beta)U(\xi) + 2\sigma(U(\xi))^3 = 0. \qquad (4.3)$$

Following the Eq. (2.4), it is easy to deduce from (4.3) that the expression of the function $F(U)$ reads

$$F(U) = \sqrt{\frac{\alpha + \beta^2 + \varepsilon\alpha\beta}{k^2(\lambda^2 + \varepsilon\lambda)}U^2 - \frac{\sigma}{k^2(\lambda^2 + \varepsilon\lambda)}U^4}, \qquad (4.4)$$

or

$$F(U)=\sqrt{c_2}U\sqrt{1-c_1U^2}, \qquad (4.5)$$

where

$$c_1=\frac{\sigma}{\alpha+\beta^2+\varepsilon\alpha\beta} \text{ and } c_2=\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{k^2(\lambda^2+\varepsilon\lambda)}.$$

Then using (2.4) transformation, the solution of the Eq. (4.3) is obtained as

$$U(\xi)=\frac{\pm 1}{\sqrt{|c_1|}}\frac{1}{\cos(\sqrt{|c_2|}(\xi-\xi_0))}. \text{ if } c_1<0 \text{ and } c_2<0,$$

$$U(\xi)=\frac{\pm 1}{\sqrt{|c_1|}}\frac{1}{\sin(\sqrt{|c_2|}(\xi-\xi_0))}. \text{ if } c_1>0 \text{ and } c_2<0, \qquad (4.6)$$

$$U(\xi)=\frac{\pm 1}{\sqrt{|c_1|}}\frac{1}{\cosh(\sqrt{|c_2|}(\xi-\xi_0))}. \text{ if } c_1>0 \text{ and } c_2>0,$$

$$U(\xi)=\frac{\pm 1}{\sqrt{|c_1|}}\frac{1}{\sinh(\sqrt{|c_2|}(\xi-\xi_0))}. \text{ if } c_1<0 \text{ and } c_2>0.$$

We obtain the following hyperbolic solutions

$$u_1(x,t)=-e^{i(\alpha x-\beta t)}\sqrt{\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{\sigma}}\, i\csc h(\sqrt{\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{k^2(\lambda^2+\varepsilon\lambda)}}k(x-t\lambda)), \qquad (4.7)$$

$$u_2(x,t)=e^{i(\alpha x-\beta t)}\sqrt{\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{\sigma}}\sec h(\sqrt{\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{k^2(\lambda^2+\varepsilon\lambda)}}k(x-t\lambda)), \qquad (4.8)$$

It is easy to see that Solutions (4.7) and (4.8) can reduce to periodic solutions as follows:

$$u_3(x,t)=e^{i(\alpha x-\beta t)}\sqrt{\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{\sigma}}\csc(\sqrt{-\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{k^2(\lambda^2+\varepsilon\lambda)}}k(x-t\lambda)), \qquad (4.9)$$

$$u_4(x,t)=e^{i(\alpha x-\beta t)}\sqrt{-\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{\sigma}}\sec(\sqrt{-\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{k^2(\lambda^2+\varepsilon\lambda)}}k(x-t\lambda)), \qquad (4.10)$$

Comparing our results and Peng's results [26] with Taghizadeh's results [27] then it can be seen that the results are the same.

V. COUPLED HIGGS FIELD EQUATION

In this section we study the Coupled Higgs field equations [27, 29]

$$u_{tt}-u_{xx}-\alpha u+\beta|u|^2u-2uv=0,$$
$$v_{tt}+v_{xx}-\beta(|u|^2)_{xx}=0. \qquad (5.1)$$

describing a system of conserved scalar nucleons interacting with a neutral scalar meson. Here, real constant v represents a complex scalar nucleon field and u is a real scalar meson field. Eq. (5.1) is the coupled nonlinear Klein Gordon equation for $\alpha<0$, $\beta<0$ and is the coupled Higgs field equation for $\alpha>0$, $\beta>0$. The existence of N-soliton solutions for Eq. (5.1) was shown by Hirota's bilinear method [30]. Recently, a class of traveling waves of the coupled Higgs equation has been derived by one of the authors [31].

By making the transformations

$$u(x,t)=e^{i\theta}U(\xi), \qquad \theta=kx+wt,$$
$$\xi=x+ct, \qquad \mathrm{v}(x,t)=\mathrm{V}(\xi)$$

where k,w and c are constants and $U(\xi)$, $\mathrm{V}(\xi)$ are real functions we have a relation $k=wc$. The System (5.1) is carried to a system of ODEs

$$(w^2(c^2-1)-\alpha)U(\xi)+(c^2-1)U''(\xi)+\beta U(\xi)^3-2U(\xi)V(\xi)=0,$$

$$(c^2+1)V''(\xi)-\beta(U(\xi)^2)''=0. \qquad (5.2)$$

Integrating second equation in (5.2) twice with respect to ξ, we have:

$$V(\xi)=\frac{R+\beta U(\xi)^2}{c^2+1} \qquad (5.3)$$

where R is the second integration constant and the first one is taken to zero. Substituting Eq. (5.3) into second equation in (5.2) yields:

$$(w^2(c^2-1)-\alpha-\frac{2R}{c^2+1})U(\xi)+(c^2-1)U''(\xi)+\beta(1-\frac{2}{c^2+1})U(\xi)^3=0. \qquad (5.4)$$

Following the Eq. (2.4), it is easy to deduce from (5.4) that the expression of the function $F(U)$ reads:

$$F(U)=\sqrt{(w^2-\frac{\alpha}{c^2-1}-\frac{2R}{c^4-1})U^2+\frac{\beta}{2(c^2+1)}U^4} \qquad (5.5)$$

Then setting:

$$U_\xi=\frac{dU}{d\xi}=F(U) \qquad (5.6)$$

We get the constants of integration to zero and we can obtain the result below:

$$U(\xi)=\frac{\pm\sqrt{|c_1|}}{\sinh(\sqrt{|c_2|}(\xi-\xi_0))}$$

and

$$V(\xi)=\frac{1}{(c^2+1)}\left(R+\frac{\pm\sqrt{|c_1|}}{\left|\sinh(\sqrt{|c_2|}(\xi-\xi_0))\right|^2}\right), \qquad (5.7)$$

$$U(\xi)=\frac{\pm\sqrt{-|c_1|}}{\sin(\sqrt{-|c_2|}(\xi-\xi_0))}$$

and

$$V(\xi)=\frac{1}{(c^2+1)}\left(R+\frac{\pm\sqrt{-|c_1|}}{\left|\sin(\sqrt{-|c_2|}(\xi-\xi_0))\right|^2}\right), \qquad (5.8)$$

$$U(\xi)=\frac{\pm\sqrt{|c_1|}}{\cosh(\sqrt{-|c_2|}(\xi-\xi_0))}$$

and

$$V(\xi)=\frac{1}{(c^2+1)}\left(R+\frac{\pm\sqrt{|c_1|}}{\left|\cosh(\sqrt{-|c_2|}(\xi-\xi_0))\right|^2}\right), \quad (5.9)$$

$$U(\xi)=\frac{\pm\sqrt{|c_1|}}{\cos(\sqrt{-|c_2|}(\xi-\xi_0))}$$ and

$$V(\xi)=\frac{1}{(c^2+1)}\left(R+\frac{\pm\sqrt{|c_1|}}{\left|\cos(\sqrt{-|c_2|}(\xi-\xi_0))\right|^2}\right), \quad (5.10)$$

where $c_1=\frac{2(c^2+1)c_2}{\beta}$, $c_2=w^2-\frac{\alpha}{c^2-1}-\frac{2R}{c^4-1}$.

We obtain the following hyperbolic solutions

$$u_1(x,t)=-\sqrt{\frac{2w^2(c^2+1)}{\beta}-\frac{2(c^2+1)\alpha}{(c^2-1)\beta}-\frac{4R}{(c^2-1)\beta}}e^{i(kx+wt)}\csc h(\sqrt{w^2-\frac{\alpha}{c^2-1}-\frac{2R}{c^4-1}}(x+ct)), \quad (5.11)$$

$$v_1(x,t)=\frac{R}{c^2+1}+(2w^2-\frac{2\alpha}{c^2-1}-\frac{4R}{c^4-1})e^{2i(kx+wt)}\csc h^2(\sqrt{w^2-\frac{\alpha}{c^2-1}-\frac{2R}{c^4-1}}(x+ct)), \quad (5.12)$$

$$u_2(x,t)=-\sqrt{\frac{2\alpha(c^2+1)}{(c^2-1)\beta}-\frac{2w^2(c^2+1)}{\beta}+\frac{4R}{(c^2-1)\beta}}e^{i(kx+wt)}\sec h(\sqrt{w^2-\frac{\alpha}{c^2-1}-\frac{2R}{c^4-1}}(x+ct)), \quad (5.13)$$

$$v_2(x,t)=\frac{R}{c^2+1}+(\frac{2\alpha}{c^2-1}-2w^2+\frac{4R}{c^4-1})e^{2i(kx+wt)}\sec h^2(\sqrt{w^2-\frac{\alpha}{c^2-1}-\frac{2R}{c^4-1}}(x+ct)), \quad (5.14)$$

It is easy to see that Solutions (5.11)-(5.14) can reduce to periodic solutions as follows:

$$u_3(x,t)=-\sqrt{\frac{2\alpha(c^2+1)}{(c^2-1)\beta}-\frac{2w^2(c^2+1)}{\beta}+\frac{4R}{(c^2-1)\beta}}e^{i(kx+wt)}\csc(\sqrt{\frac{\alpha}{c^2-1}-w^2+\frac{2R}{c^4-1}}(x+ct)), \quad (5.15)$$

$$v_3(x,t)=\frac{R}{c^2+1}+(\frac{2\alpha}{c^2-1}-2w^2+\frac{4R}{c^4-1})e^{2i(kx+wt)}\csc^2(\sqrt{\frac{\alpha}{c^2-1}+\frac{2R}{c^4-1}-w^2}(x+ct)), \quad (5.16)$$

$$u_4(x,t)=-\sqrt{\frac{2\alpha(c^2+1)}{(c^2-1)\beta}-\frac{2w^2(c^2+1)}{\beta}+\frac{4R}{(c^2-1)\beta}}e^{i(kx+wt)}\sec(\sqrt{\frac{\alpha}{c^2-1}-w^2+\frac{2R}{c^4-1}}(x+ct)), \quad (5.17)$$

$$v_4(x,t)=\frac{R}{c^2+1}+(\frac{2\alpha}{c^2-1}-2w^2+\frac{4R}{c^4-1})e^{2i(kx+wt)}\sec^2(\sqrt{\frac{\alpha}{c^2-1}+\frac{2R}{c^4-1}-w^2}(x+ct)), \quad (5.18)$$

Comparing our results and Bekir.s results [32] with Taghizadeh's results [27] then it can be seen that the results are the same.

Remark 1: There is no a general formula for solutions of complex nonlinear equations. Although a complex nonlinear equation may well have a solution involving an arbitrary parameter, there may also be other solutions. We attempted to derive as many solutions to the Eqs. (3.1), (4.1) and (5.1) as possible by our methods. The obtained solutions might make good physical sense in applications.

Remark 2: There is no a universal method for complex nonlinear evolution equations. Each of the existing methods presented in literature for solving complex nonlinear evolution equations has some advantages and disadvantages. However, there are some reasons to select the functional variable method over the others. (i) The function method provides hyperbolic and periodic wave solutions by setting the parameters as special values. (ii) This method has many advantages: it is direct, simple, straightforward, effective and concise.

Remark 3: All the solutions reported in this paper have been verified with Maple by putting them back into the original (3.1), (4.1) and (5.1).

VI. CONCLUSIONS

We observed that the functional variable method could be applied to NEEs which could be converted to a second order ODE through the traveling wave transformation. From our results, we can see that the technique used in this paper is very effective and can be steadily applied to nonlinear problems. On the other hand, it can be applied some non-integrable equations, arising in applied mathematics. As a result, many exact solutions are obtained with the help of symbolic system Maple including soliton solutions presented by hyperbolic functions sech and cosech, periodic solutions presented by sec and cosec and rational solutions. The functional variable method was successfully used to establish exact solutions. Thus, we conclude that the proposed method can be extended to solve the nonlinear problems which arise in the theory of solitons and other areas.

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